Fix notations: $\mathfrak{g}$ finite-dim simple Lie Alg

$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[[t]] \oplus \mathbb{C}c$ the affine Lie Alg

This note is basically a copy of Sam Jerald's IFS note with my comments together with some connections with previous talks.

where c central and

$[x \otimes f, y \otimes g] = [x,y] \otimes fg + (x,y) \operatorname{Res}_{t=0} f dg \cdot c$

$\downarrow$ may write $x(f)$

Set $\hat{\mathfrak{g}}_+ := \mathfrak{g}[[t]] := \mathfrak{g} \otimes t\,\mathbb{C}[[t]]$

$\hat{\mathfrak{g}}_- := \mathfrak{g}[t^{-1}] := \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]$

then $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_+ \oplus \mathfrak{g} \oplus \hat{\mathfrak{g}}_- \oplus \mathbb{C}c$

* Root system, wgt lattice, Weyl gp those things also exist in $\hat{\mathfrak{g}}$, denote them by the same symbols with $\wedge$

Set θ to be hgt root. $\theta^\vee$ is the dual coroot

Def: take $\lambda \in P^+$, $l \in \mathbb{Z}_+$, we say λ is of level l if $\langle \lambda, \theta^\vee \rangle \leq l$.

Set $P_l^+ := \{ \lambda \in P^+ \mid \langle \lambda, \theta^\vee \rangle \leq l \}$

Aim: we want to define fusion rule on P_l^+

This case might be more interesting, because it's more computable, and physicists only care about this case

$\uparrow$

unitary reps of affine Lie algs must have non-negative integer level.

What's the difference between $\kappa \in \mathbb{Q}_{<0}$ and $l \in \mathbb{Z}_+$?

the answer is finiteness (and semisimplicity)

this is why easier

Prop: (16.3.3. Chari & Pressley)

1) $\mathcal{O}_\kappa$ is semisimple if $\kappa \notin \mathbb{Q}$

2) Every obj of $\mathcal{O}_\kappa$ has a composition series of finite length if $\kappa \notin \mathbb{Q}_{\geq 0}$ (In particular, $l \in \mathbb{Z}_+$)

crucial for existence of $\dot{\otimes}$ we defined

To be explicit, recall $W := \bigsqcup_{k \geq 1} Z^k$

$$Z = \mathrm{Hom}_{\mathbb{C}}\Big(\bigotimes_{s \leq s_0} V_s, \mathbb{C}\Big), \quad Z^k \subset Z \ (G_k \cdot \varphi = 0)$$

G_k spanned by $x_1(f_1) \cdots x_k(f_k)$ s.t. $(f_i)_{s_0} \in z\mathbb{C}[[z]]$

"the tricky point" of showing $W \in \mathcal{O}_\kappa$ is that W is f.g, where we should use $\kappa \notin \mathbb{Q}_{\geq 0}$

for $l \in \mathbb{Z}_+$, we turn to $\mathcal{O}^{int}$ by considering integral modules then we also have the finite-length property and define $\dot{\otimes}$ similarly, (easier)

Recall the def of generalized Verma module:

take $\lambda \in P^+$, $M(\lambda, l) := U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{p}})} I_\lambda(V(\lambda))$

Here $\hat{\mathfrak{p}} = \hat{\mathfrak{g}}_+ \oplus \mathbb{C}c \oplus \mathfrak{g}$, i.e. $\hat{\mathfrak{g}} = \hat{\mathfrak{p}} \oplus \hat{\mathfrak{g}}_-$.

$I_\lambda(V(\lambda))$ has $U(\hat{\mathfrak{p}})$-structure given by:

$$\hat{\mathfrak{g}}_+ \cdot V(\lambda) = 0$$
$$c \cdot V(\lambda) = l.\mathrm{Id} \cdot V(\lambda)$$

It follows $M(\lambda, l)$ has unique maximal submodule $N(\lambda, l)$ generated by $x_\theta(t^{-1})^{l - \lambda(\theta^\vee) + 1} \cdot v_\lambda$

$\in \hat{\mathfrak{g}}_\theta$ $\qquad$ $\neq 0 \in F(V_\lambda)_\lambda$

Let $H(\lambda) := H(\lambda, l) := M(\lambda, l) / N(\lambda, l)$

Thm: $H(\lambda, l)$ is integrable, irreducible highest wgt $\hat{\mathfrak{g}}$-mod. Conversely any integrable highest wgt $\hat{\mathfrak{g}}$-mod is iso to some $H(\lambda, l)$

#1. Space of Vacua and Covacua

↑ conformal blocks — ↑ dual conformal blocks

$\mathfrak{g}$, $l \in \mathbb{Z}_+$, $\Sigma = \Sigma_g$ ($= \mathbb{P}^1$ in our lecture)

a smooth connected proj curve over $\mathbb{C}$ of genus g

Indeed, the independence of fusion rules with the chosen pts follows from geometric structure since we use dimension to define!

Take $\vec{P} = (P_1, \cdots, P_s)$ where $s \geq 1$, $P_i \in \Sigma$

as mentioned in Sam's talks, $s=3$ is most important.

$\vec{\lambda} = (\lambda_1, \cdots, \lambda_s)$ $\lambda_i \in P_l^+$

Let $\mathcal{O}(U)$ be regular functions on U, $\mathfrak{g}(U) = \mathfrak{g} \otimes \mathcal{O}(U)$ take $U = \Sigma - \vec{P}$, for $\forall P_i \in \vec{P}$, choose a local coordinate t_i, let $f_{P_i} \in \mathbb{C}[[t_i]]$ be the local Laurent expansion of f near P_i, hence we have an alg homo:

$$\mathcal{O}(\Sigma - \vec{P}) \to \mathbb{C}[[t_i]]$$

$\mathfrak{g}_{\mathfrak{S}, \text{glob}}$

induces Lie alg homo: $\mathfrak{g}(\Sigma - \vec{P}) \to \mathfrak{g}[[t_i]]$

Set $H(\vec{\lambda}) := H(\lambda_1) \otimes \cdots \otimes H(\lambda_s)$

define $\mathfrak{g}(\Sigma - \vec{p})$-action on it (like before!)

$$x(f).(v_1 \otimes \cdots \otimes v_s) := \sum_{i=1}^{s} v_1 \otimes \cdots \otimes x(f_{p_i}) v_i \otimes \cdots \otimes v_s$$

Def: The space of Vacua is given by:

$$V_\Sigma^\dagger(\vec{p}, \vec{\lambda}) := \mathrm{Hom}_{\mathfrak{g}(\Sigma - \vec{p})}(H(\vec{\lambda}), \underset{\substack{\uparrow \\ \text{trivial-mod}}}{\mathbb{C}})$$

The space of Covacua is given by:

$$V_\Sigma(\vec{p}, \vec{\lambda}) := [H(\vec{\lambda})]_{\mathfrak{g}(\Sigma - \vec{p})}$$
$$= H(\vec{\lambda}) / \mathfrak{g}(\Sigma - \vec{p}).H(\vec{\lambda})$$

i.e. the space of coinvariants. we also define this in $\kappa \in \mathbb{Q}_{<0}$ case, but use $\mathfrak{g}_{s.glob}$ notation

LM: 1) $V_\Sigma^\dagger(\vec{p}, \vec{\lambda}) \cong V_\Sigma(\vec{p}, \vec{\lambda})^*$ (Duality)

2) $\dim_{\mathbb{C}} V_\Sigma(\vec{p}, \vec{\lambda}) < \infty$

Pf: 1) $V_\Sigma^\dagger(\vec{p}, \vec{\lambda}) \overset{①}{=} \mathrm{Hom}_{\mathfrak{g}(\Sigma - \vec{p})}(H(\vec{\lambda}), \mathbb{C})$

$$\overset{②}{=} \mathrm{Hom}_{\mathbb{C}}(H(\vec{\lambda}), \mathbb{C})^{\mathfrak{g}(\Sigma - \vec{p})}$$
$$\overset{③}{=} \mathrm{Hom}_{\mathbb{C}}(H(\vec{\lambda}) / \mathfrak{g}(\Sigma - \vec{p}).H(\vec{\lambda}), \mathbb{C})$$
$$= V_\Sigma(\vec{p}, \vec{\lambda})^*$$

① $f(a.x) = a.f(x) = f(x)$

② $(a.f)(x) = f(-a.x) = f(x)$

③ $f(\mathfrak{g}.x) = 0$

2) hard, but we know this from previous lecture as well!

Recall $V(\lambda)$ $\lambda \in P^+$ irr $\mathfrak{g}$-mod

$\Rightarrow V(\lambda)^*$ a $\mathfrak{g}$-mod with hgw $\overset{\lambda^*}{\overset{\shortparallel}{-w_0\lambda}}$

<u>LM:</u> $\lambda \mapsto \lambda^*$ preserves P^+_l

<u>Pf</u>: $\langle \lambda^*, \theta^\vee \rangle = \langle -w_0\lambda, \theta^\vee \rangle$

$= \langle \lambda, -w_0\theta^\vee \rangle$

$= \langle \lambda, \theta^\vee \rangle \leq l$

note $w_0\theta^\vee = -\theta^\vee$ $\square$

<u>Prop</u>: Let $\vec{\lambda}^* = (\lambda_1^*, \cdots \lambda_s^*)$

then $V_\Sigma(\vec{p}, \vec{\lambda}) \cong V_\Sigma(\vec{p}, \vec{\lambda}^*)$

<u>Pf of Sketch</u>: For any auto σ of $\mathfrak{g}$, we can define an endofunctor of $\mathrm{Rep}(\mathfrak{g})$ by compose σ.

by extending σ to $\hat{\sigma} \in \mathrm{Aut}(\hat{\mathfrak{g}})$, we can show for $\pi_\lambda : \hat{\mathfrak{g}} \to \mathrm{End}(H_\lambda)$, $\pi_\lambda \circ \hat{\sigma} \cong \pi_{\lambda^*}$

We can do this to each factor.

Propigation of Vacua (Make it easier to compute!)

Let $\vec{p} = (p_1, \cdots, p_s) \in \Sigma \ni \vec{q} = (q_1, \cdots, q_t)$

and every point is dinstinct

Also Fix $\vec{\lambda} = (\lambda_1, \cdots \lambda_s) \subset P_l^t \ni \vec{\mu} = (\mu_1, \cdots, \mu_t)$

Set $V(\vec{\mu}) = V(\mu_1) \otimes \cdots \otimes V(\mu_t)$, it's $\mathfrak{g}$-mod

We want to define a $\mathfrak{g}(\Sigma - \vec{p})$-action on $V(\vec{\mu})$

(this is different with $k \in \mathbb{Q}_{<0}$ case)

$$x(f) . (v_1 \otimes \cdots \otimes v_t) := \sum_{i=1}^{t} v_1 \otimes \cdots \otimes \underset{\substack{\uparrow \\ \text{evaluation at } q_i}}{f(q_i)} x . v_i \otimes \cdots \otimes v_t$$

this is valid because $f \in \mathcal{O}(\Sigma - \vec{p})$

and $q_i \bar{\in} \vec{p}$.

Hence we can define a $\mathfrak{g}(\Sigma - \vec{p})$-action

on $\underset{\substack{\uparrow \\ \text{Laurent Exp}}}{H(\vec{\lambda})} \otimes \underset{\substack{\uparrow \\ \text{evaluation}}}{V(\vec{\mu})}$

Thm: The natural map: covacua $\leftrightarrow$ vacua

$$[H(\vec{\lambda}) \otimes V(\vec{\mu})]_{\mathfrak{g}(\Sigma - \vec{p})} \xrightarrow{\sim} V_\Sigma(\vec{p} \sqcup \vec{q}, \vec{\lambda} \sqcup \vec{\mu})$$

induced from $\mathfrak{g}(\Sigma - \vec{p})$-mod homo $V(\mu_i) \hookrightarrow H(\mu_i)$

is an iso

Pf: No but focus on applications.

Corollary: (Propigation of Vacua)

Take a point $\vec{q} \in \Sigma - \vec{p}$, we have

(a) $V_\Sigma(\vec{p}, \vec{\lambda}) \cong V_\Sigma(\vec{p} \sqcup \{q\}, \vec{\lambda} \sqcup \{0\})$

add additional punctures with wgt 0 does not change vacua

(b) $V_\Sigma(\vec{p}, \vec{\lambda}) \cong [H(0) \otimes V(\vec{\lambda})]_{\mathfrak{g}(\Sigma - \{q\})}$

replace s-punctures by 1-punctures

Pf: Make use of thm

(a) $V_\Sigma(\vec{p} \sqcup \{q\}, \vec{\lambda} \sqcup \{0\}) \cong [H(\vec{\lambda}) \otimes \underset{\text{trivial}}{V(0)}]_{\mathfrak{g}(\Sigma - \vec{p})}$

$\cong [H(\vec{\lambda})]_{\mathfrak{g}(\Sigma - \vec{p})}$

$\overset{\text{by def}}{=} V_\Sigma(\vec{p}, \vec{\lambda})$

(b) $[H(0) \otimes V(\vec{\lambda})]_{\mathfrak{g}(\Sigma - \{q\})} \cong V_\Sigma(\{q\} \sqcup \vec{p}, \{0\} \sqcup \vec{\lambda})$

$\overset{\text{by (a)}}{\cong} V_\Sigma(\vec{p}, \vec{\lambda})$ $\square$

Now. $\Sigma = \mathbb{P}^1 = \mathbb{A}^1 \sqcup \{\infty\}$, note $\mathbb{A}^1 = \mathbb{C}$

fix $\vec{p} = (p_1, \cdots p_s) \in (\mathbb{A}^1)^s$ (note by some automorphism of $\mathbb{P}^1$, we can always do this)

fix $\vec{\lambda} = (\lambda_1, \cdots \lambda_s) \in (P_\ell^+)^s$

hgt root θ, hgt coroot $\theta^\vee$, take $X_\theta (= E_\theta) \in \mathfrak{g}_\theta$

$X_{-\theta} (= F_\theta) \in \mathfrak{g}_{-\theta}$

such that $[X_\theta, X_{-\theta}] = \theta^\vee$

we know $\langle X_\theta, X_{-\theta}, \theta^\vee \rangle \cong sl_2$

$X_\theta \mapsto e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$X_{-\theta} \mapsto f = \begin{bmatrix}0&0\\1&0\end{bmatrix}$$

$$\theta^\vee \longmapsto h = \begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

denote $sl_2(\theta) = \langle X_\theta, X_{-\theta}, \theta^\vee \rangle$

For $\forall\, V(\vec{\lambda}) = V(\lambda_1)\otimes \cdots \otimes V(\lambda_s)$ $\mathfrak{g}$-mod

we define operator φ (evaluation operator)

by $\varphi(v_1\otimes \cdots \otimes v_s) = \sum_{i=1}^{s} p_i \underbrace{v_1\otimes \cdots \otimes X_\theta v_i \otimes \cdots \otimes v_s}$

$$(X_\theta \otimes t)_{(p_i)} v_i = t(p_i)\, X_\theta v_i = p_i\, X_\theta v_i$$

Thm: a) $V_{|\vec{p}|}(\vec{p}, \vec{\lambda}) \cong V(\vec{\lambda}) \big/ \left(\mathfrak{g}\cdot V(\vec{\lambda}) + \mathrm{Im}\,\varphi^{l+1} \right)$

in particular, if $l \gg 0$, $\mathrm{Im}\,\varphi^{l+1} = 0$

then $V_{|p|}(\vec{p}, \vec{\lambda}) \cong V(\vec{\lambda}) \big/ \mathfrak{g}. V(\vec{\lambda})$

Recall when $\kappa \in \mathbb{Q}_{<0}$, we have similar

for $H_{\vec{\lambda}}$

result, at that thm, there is no $\mathrm{Im}\,\varphi^{l+1}$ term.

this implies, $l \gg 0$ is important. Indeed,

Finkelberg's result $\overline{\mathrm{til}} \cong \mathcal{O}_{int}$ has

that assumption. ($l \geq \check{h} + 3$)

b) $V^{+}_{|\vec{p}|}(\vec{p}, \vec{\lambda}) \cong \{ \mathfrak{g}\text{-mod maps } f : V(\vec{\lambda}) \to \mathbb{C} \text{ such that } f \circ \varphi^{l+1} = 0 \}$

this follows from a) and $V^{+}_{|\vec{p}|}(\vec{p}, \vec{\lambda}) = V_{|\vec{p}|}(\vec{p}, \vec{\lambda})^{*}$

Pf: I am not going to talk about this

It relies on the fact we choose pts not ∞

and the description of max submodule of $M(\lambda, l)$

Eg: Take ($\Sigma = \mathbb{P}^1$) , $\mathfrak{g} = sl_2$, $s = 3$

For $n \in \mathbb{N}_+$, ω fundamental wgt $\Rightarrow P^+ = \mathbb{Z}_+ \omega$

Set $V(n) = V(n\omega)$, $\dim V(n) = n+1$, irr sl_2-rep

Prop: a) For $(n_1, n_2, n_3) \in \mathbb{Z}_+^3$, the space

$V(\vec{\lambda}) / sl_2 \cdot V(\vec{\lambda}) = \mathrm{Hom}_{sl_2}(V(\vec{\lambda}), \mathbb{C})$ is at most 1-dim

and it's 1-dim iff $\begin{bmatrix} n_1+n_2+n_3 \in 2\mathbb{Z}_+ \\ \text{sum of any two of them} \\ \text{must} \geq \text{the third} \end{bmatrix}$

b) For $(n_1, n_2, n_3) \in \{0, 1, \cdots, \ell\}^3$

$V^+_{\mathbb{P}^1}((P_1, P_2, P_3), (n_1\omega, n_2\omega, n_3\omega))$ is at most one-dim. Moreover, it's 1-dim iff holds

& $n_1+n_2+n_3 \leq 2\ell$

Pf: For a) Recall Clebsch-Gordan Formula

i.e. $V(n) \otimes V(m) = \sum_{i=0}^{q} V(m+n-2i)$

where $q = \min(m, n)$

We may suppose $n_1 \leq n_2 \leq n_3$

then $V(n_1) \otimes V(n_2) \otimes V(n_3)$

$$= \bigoplus_{i=0}^{n_1} V(n_1+n_2-i) \otimes V(n_3)$$

$$= \bigoplus_{i=0}^{n_1} \bigoplus_{j=0}^{\min(n_1+n_2-i,\, n_3)} V(n_1+n_2+n_3-i-2j)$$

if $n_1+n_2 < n_3$, then $n_1+n_2+n_3-i-2j$

$$\geq n_1+n_2+n_3-i-2(n_1+n_2-i)$$

$$= n_3-n_1-n_2+i > 0$$

However, in this case, the coinvariance is zero

indeed, there can be nonzero if there is trivial

module. there can be at most 1-such term

which is just [ ]

For b) $V^{+}_{|p|}((p_1, p_2, p_3), (n_1, n_2, n_3))$

$$\cong \{ f \in \underbrace{\mathrm{Hom}_{sl_2}(V(\vec{\lambda}), \mathbb{C})}_{\text{at most 1-dim by a)}} \mid f \circ \varphi^{(+)} = 0 \}$$

if [ ] not hold it's 0

assume [ ] holds

Take standard basis of $V(1) \cong \mathbb{C}^2$ $\{e_1, e_2\}$

$$\mathrm{Hom}_{\mathbb{C}}(V(\vec{\lambda}), \mathbb{C}) \xleftrightarrow[\text{why?}]{\text{bijection}} \text{poly of deg} \leq n_i \text{ in three variables}$$

$\vec{\lambda} = (n_1, n_2, n_3)$

$$f \longmapsto P(x_1, x_2, x_3) = f\left((e_1 + x_1 e_2)^{\otimes n_1} \otimes (e_1 + x_2 e_2)^{\otimes n_2} \otimes (e_1 + x_3 e_2)^{\otimes n_3}\right)$$

Let $n = \frac{n_1 + n_2 + n_3}{2}$, the 1-dim subspace

$\mathrm{Hom}_{sl_2}(V(\vec{\lambda}), \mathbb{C})$ corresponds to the poly

(up to scaling) $P_0(x_1, x_2, x_3) = (x_2 - x_3)^{n - n_1}(x_3 - x_1)^{n - n_2}(x_1 - x_2)^{n - n_3}$

Use $x_\theta \otimes 1 \otimes 1$ acts by derivation w.r.t x_1

$1 \otimes x_\theta \otimes 1$ x_2

$1 \otimes 1 \otimes x_\theta$ x_3

$P_0 \circ \varphi^m$ corresponds to $\frac{\varepsilon^m}{m!}$ when expand

$P_0(x_1 + p_1\varepsilon, x_2 + p_2\varepsilon, x_3 + p_3\varepsilon)$

$$= [x_2 - x_3 + \varepsilon(p_2 - p_3)]^{n-n_1} [x_3 - x_1 + \varepsilon(p_3 - p_1)]^{n-n_2}$$
$$[x_1 - x_2 + \varepsilon(p_1 - p_2)]^{n-n_3}$$

Note P_0 is of deg $3n - n_1 - n_2 - n_3 = n$ (by def of n)

so if $l+1 > n \Rightarrow P_0 \circ \varphi^{l+1} = 0$

i.e. $n_1 + n_2 + n_3 \leq 2l$

Conversely, if $n \geq l+1$, then expand above and use fact that (p_1, p_2, p_3) are distinct, this gives $P_0 \circ \varphi^{l+1} \neq 0$

$n_1+n_2+n_3 \leq 2l$ & $\boxed{}$

$\Updownarrow$

Hence from $V^{+}_{|P^1|}(\vec{p}, \vec{\lambda}) \cong \{f \in \mathrm{Hom}_{\mathfrak{g}}(V(\vec{\lambda}), \mathbb{C}) \mid f \circ \varphi^{l+1} = 0\}$ $\square$

Eg: If we're dealing with $\Sigma = \mathbb{P}^1$, $s = 1$ but $\mathfrak{g}$ not necessary sl_2, just as usual, we branch over $sl_2(\theta)$

Take $V(\lambda) \Rightarrow V(\lambda) = \bigoplus_{n \in \mathbb{Z}_+} V(\lambda)_{(n)}$ where $V(\lambda)_{(n)}$

is the $sl_2(\theta)$-isotypic compotent of wgt n

set $V(\vec{\lambda})_F := \bigoplus_{n_1+n_2+n_3 > 2l} V(\vec{\lambda})_{(n_1, n_2, n_3)}$

where $V(\lambda)_{(n_1, n_2, n_3)} := V(\lambda_1)_{(n_1)} \otimes V(\lambda_2)_{(n_2)} \otimes V(\lambda_3)_{(n_3)}$

<u>Thm</u>: $V^{+}_{|P^1|}(\vec{p}, \vec{\lambda}) \cong \{f \in \mathrm{Hom}_{\mathfrak{g}}(V(\vec{\lambda}), \mathbb{C}) \mid f|_{V(\vec{\lambda})_F} = 0\}$

<u>Ex</u>: ① $p \in \mathbb{A}^1$, $\lambda \in P^{+}_{l}$, $V_{|P^1|}(p, \lambda) = \begin{cases} 0 & \lambda \neq 0 \\ \mathbb{C} & \lambda = 0 \end{cases}$

② $p \neq q \in \mathbb{A}^1$, $\lambda, \mu \in P^{+}_{l}$, $V_{|P^1|}((p,q), (\lambda, \mu)) = \begin{cases} 0 & \lambda \neq \mu^* \\ \mathbb{C} & \lambda = \mu^* \end{cases}$

Fusion Rules :

Question : What's this formally?

A finite set with involution $*$ i.e. $*^2 = id_A$

Set $\mathbb{Z}_+[A] := \bigoplus_{a \in A} \mathbb{Z}_+ a$ be the free monoid generated by A. $*$ extends to $\mathbb{Z}_+[A]$ by linearity

Def: A fusion rule on A is a map $F: \mathbb{Z}_+[A] \to \mathbb{Z}$

s.t. 1) $F(0) = 1$

2) $F(a) > 0$ for some $a \in A$

3) $F(x) = F(x^*)$, for $\forall x \in \mathbb{Z}_+[A]$

4) $F(x+y) = \sum_{\lambda \in A} F(x+\lambda) F(y+\lambda^*)$ $\forall x, y \in \mathbb{Z}_+[A]$

Furthermore, it's non-degenerate if

5) $\forall a \in A$, $\exists \lambda_a \in A$ s.t. $F(a + \lambda_a) \neq 0$

Why do we need this?

Prop: Let $F: \mathbb{Z}_+[A] \to \mathbb{Z}$ be a non-deg fusion rule on A then the abelian gp $\mathbb{Z}[A] = \bigoplus_{a \in A} \mathbb{Z} a$ becomes a commutative ring with identity under the fusion-product given by:

$$a \cdot b = \sum_{\lambda \in A} F(a + b + \lambda^*) \lambda \quad \forall a, b \in A$$

(extend linearly)

Moreover, there $\exists$ a unique linear form called trace

$t: \mathbb{Z}[A] \to \mathbb{Z}$ s.t.

1) $t(a \cdot b^*) = \delta_{a,b} \quad \forall a, b \in A$

2) $t(\prod_{a\in A} a^{n_a}) = F(\sum_{a \in A} n_a a)$ $\square$

We say $(\mathbb{Z}[A], t)$ is the fusion ring associated to the fusion rule F.

Back to our world!

Def: Let $F_l: \mathbb{Z}_+[P_l^+] \to \mathbb{Z}_+$ be the following map:

1) $F(\underline{0}) = 1$ where $\underline{0}$ is zero in $\mathbb{Z}_+[P_l^+]$

2) $F_l(\lambda_1 + \cdots + \lambda_s) := \dim V_{\mathbb{P}^1}(\vec{p}, \vec{\lambda})$

for some pts $\vec{p} \in \mathbb{P}^1$ & $\vec{\lambda} = (\lambda_1, \cdots \lambda_s)$

Independence of $\vec{p}$

Thm: Let $(\Sigma, \vec{p})$ be a smooth connected s-pted proj curve $(s \geq 1)$ of genus $g \geq 0$ such that $2g-2+s>0$. Then $\forall \vec{\lambda} \in (P_l^+)^s$, $\dim_{\mathbb{C}} V_\Sigma(\vec{p}, \vec{\lambda})$ only depends on g and $\vec{\lambda}$.

In our case $\Sigma = \mathbb{P}^1$, $g=0$, for $s \geq 3$, because of thm, the independence is verified

for $s=2$ or $s=1$ we know from above Ex

$\dim V_{\mathbb{P}^1}(\vec{p}, \vec{\lambda}) = 0$ or 1 depends only on $\vec{\lambda}$

Take $\lambda^* = -w_0\lambda$, the above F gives a non-deg fusion rule

1) $F(\underset{\sim}{0}) = 1$

2) $F_l(\underset{P_l^+}{0}) = 1$ by Ex

3) $F_l(\vec{\lambda}) = \dim V_{|P|}(\vec{p}, \vec{\lambda}) = \dim V_{|P|}(\vec{p}, \vec{\lambda}^*) = F_l(\vec{\lambda}^*)$

4) Factorization thm

5) For $\lambda \in P_l^+$, $V_{|P|}((p,q), (\lambda, \lambda^*)) \overset{p\neq q}{\cong} \mathbb{C}$

hence $\forall \lambda \in P_l^+$, $\exists \mu \in P_l^+$ $F_l(\lambda + \mu) \neq 0$

Now why $s=3$ is important?

For simple Lie alg $\mathfrak{g}$, the fusion ring at level l denoted $R_l(\mathfrak{g})$, is the fusion ring $\mathbb{Z}[P_l^+]$ with F_l defined above. as $\mathbb{Z}$-mod. we may regard it as free generated by iso classes $\{[V\lambda] \mid \lambda \in P_l^+\}$

the level l fusion product is given by

$$[V\lambda] \otimes_l [V\mu] := \sum_{\eta \in P_l^+} \dim V_{|P|}(\vec{P}, (\lambda, \mu, \eta^*))[V(\eta)]$$

Eg: take sl_2 case $n = n\omega \in P_l^+$ $n^* = n$

take $n, m \leq l$

$$[V(\lambda)] \otimes V(\mu) \models [V(n)] \otimes_l [V(m)] = \sum_{i=\max(0,\, m+n-l)}^{\min(n,m)} [V(n+m-2i)]$$

Prop: For $\forall \lambda, \mu \in P_l^+$, $[V(\lambda)] \otimes_l [V(\mu)]$ is the iso class of $V(\lambda) \otimes V(\mu)$ quotient by the $\mathfrak{g}$-sub mod generated by

$$\bigoplus_{p+q+r > 2l} \left(V(\lambda)_{(p)} \otimes V(\mu)_{(q)} \right)_{(r)}$$

Coro: If $\lambda + \mu \in P_l^+$, then

$$[V(\lambda)] \otimes_l [V(\mu)] = [V(\lambda) \otimes V(\mu)]$$

i.e. fix λ, μ fusion product recover the usual tensor product for $l \gg 0$

Pf:

sl_2 case $\quad \lambda + \mu \in P_l^+ \Leftrightarrow n + m \leq l$

i.e. $n+m-l \leq 0 \Rightarrow \max(0, n+m-l) = 0$

hence no term vanish

general: The largest isotypic comp of $V(\lambda)$ $V(\mu)$ are $\lambda(\theta^\vee)$, $\mu(\theta^\vee)$ and the largest component (r) is $(\lambda+\mu)(\theta^\vee)$. If $\lambda + \mu \in P_l^+$

$$p+q+r \leq \lambda(\theta^\vee) + \mu(\theta^\vee) + (\lambda+\mu)(\theta^\vee)$$
$$\leq 2l$$

the quotient is zero $\square$